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Free surface flow due to a sink

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Two-dimensional free surface flows without waves, produced by a submerged sink in a reservoir, are computed numerically for various configurations. For a sink above the horizontal bottom of a layer of fluid, there are solutions for all values of the Froude number F greater than some particular value. However, when the fluid is sufficiently deep, there is an additional solution for one special value of $F < 1$. The results for a sink at the vertex of a sloping bottom, treated by Craya and by Hocking, and for a sink in fluid of infinite depth, treated by Tuck & Vanden-Broeck, are confirmed and extended. In particular it is shown that as the bottom tends to the horizontal, the solution for a sink at the vertex of a sloping bottom approaches a solution for a horizontal bottom with $F = 1$. However solutions are found for all values of the Froude number $F \geqslant 1$ for a sink on a horizontal bottom.

1. Introduction

When fluid is withdrawn from a reservoir by a sink of strength Q at depth h, the surface above the sink may be drawn down, as in figures 1 and 6. An exact solution of this type was found by Craya (1949) (also in Yih 1965, pp. 124–126) when the bottom sloped downward at the angle $\beta = \frac{1}{3}\pi$ from the vertical, and a numerical solution was found by Tuck & Vanden-Broeck (1984) for $\beta = 0$. Then Hocking (1985) obtained numerical solutions for a sequence of angles ranging from 0 to $\frac{1}{2}\pi$. In each case there was just one solution without waves. Moreover Collings (1986) calculated solutions for infinite Froude number when $\beta = \frac{1}{2}\pi$. We have recomputed these two-dimensional flows and confirmed their solutions.

For a horizontal bottom, however, we have found solutions for all values of F greater than some particular value. When the fluid is sufficiently deep, there is an additional solution for one special value of $F < 1$ We were led to look for these solutions by our experience with other free surface flows with gravity, such as flows over weirs in channels and flows around lips of teapot spouts. In those cases we found that in fluid of infinite depth there was a flow only for a special value of the appropriate Froude number. This kind of flow also occurred in fluid of finite depth, but in addition there were solutions for all Froude numbers greater than some particular value. Our present results show that this is also the case for free surface flows produced by sinks.

In §2 we compute wave-free flows produced by a sink in a liquid layer of finite depth with a horizontal bottom. We find that the sink strength Q and the Froude number F are determined by the sink depth h and the distance W from the sink to the bottom.

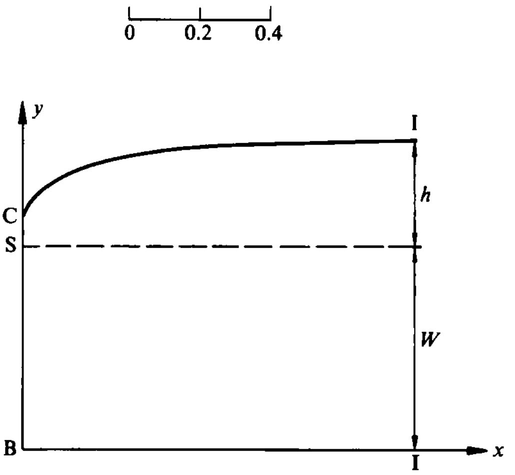

FIGURE 1. Vertical section of a reservoir with a sink at S on the vertical wall BSC, a horizontal bottom BI and a free surface CI. The height of the sink above the bottom is W, and its depth below the free surface at infinity is h. The x-axis is along the bottom and the y-axis is along the wall. This figure is an actual computed surface profile for $h/W = 0.5$. The Froude number F is 0.44. The vertical scale is the same as the horizontal scale.

The results are discussed in §3. It is pointed out that the solutions are subcritical for $0 < h/W < 0.70$ and supercritical for $0.70 < h/W < 0.76$, while no solutions are found for $h/W > 0.76$. Additional supercritical solutions are constructed in §4, and it is indicated that there are additional subcritical solutions with waves. In §5, we find solutions for a sink at the corner on a sloping bottom, which were mentioned in the first paragraph of this section.

2. Sink above a horizontal bottom

Let us consider the flow in the region of the z-plane shown in figure 1. At the distance W from a horizontal bottom there is a sink with strength $2Q$. The bottom BI is a streamline on which we require that the stream function $\psi(x, y) = Q$. The portion BS of the vertical wall, where $x = 0, 0 < y < W$, is part of the same streamline. The portion SC of the vertical wall, where $x = 0$, $W < y < y_c$, is part of another streamline on which $\psi(x, y) = 0$. That streamline continues along the free surface CI, which leaves the wall tangentially at some point C which is to be determined.

We denote by h the depth of the sink below the level of the free surface at infinity. We also choose the unit of length and the unit of velocity so that $Q = 1$ and $g = 1$. Then we introduce the complex velocity potential $f(z) = \phi(x, y) + i\psi(x, y)$. In the f-plane, the flow region is the strip $0 < \psi < 1$ with the streamline ICS on $\psi = 0$ and the streamline IBS on $\psi = 1$. We map it onto the lower half of the unit circle of the auxiliary t-plane by the transformation

$$f = -\frac{1}{\pi} \log \frac{4t}{(t+1)^2} \tag{2.1}$$

(see figure 2).

Next we consider the complex velocity $\zeta(t) = u - iv$ as a function of t in the semicircle in figure 2. Here $u = \phi_x$ and $v = \phi_y$ are the velocity components. Since there

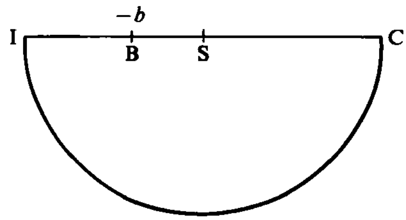

FIGURE 2. The image of the flow region in the t-plane is the lower half of the unit circle with I at $t = -1$, B at $t = -b$, S at $t = 0$ and C at $t = 1$.

is a sink of strength -2 at $t = 0$ and a right-angled corner at $x = y = 0$, ζ must have singularities at these two points and be regular elsewhere. The appropriate singularities are proportional to t^{-1} at $t = 0$ and to $(t+b)^{\frac{1}{2}}$ at $t = -b$, where $-b$ is the image of the corner B in the t-plane (see figure 2). Therefore we write $\zeta(t)$ in the form

$$\zeta(t) = \mathrm{i}(t+b)^{\frac{1}{2}}t^{-1}\exp\left(\sum_{n=0}^{\infty} a_n t^n\right). \tag{2.2}$$

The coefficients a_n are to be found.

On the wall from B to C, $u = 0$ so $\zeta(t)$ must be imaginary for $-b < t < 1$. In addition on the bottom from B to I, $v = 0$, so $\zeta(t)$ must be real for $-1 < t < -b$. Both of these conditions are satisfied by (2.2) if we choose all the a_n to be real.

On the free surface from C to I the pressure is assumed to be constant. By using the Bernoulli equation we can write this condition as $|\zeta|^2 + 2y = \text{constant}$. In the t-plane it must hold on the circular arc $t = \mathrm{e}^{\mathrm{i}\sigma}$, $0 \geqslant \sigma \geqslant -\pi$. Differentiating this condition with respect to σ yields $\partial|\zeta|^2/\partial\sigma + 2\partial y/\partial\sigma = 0$. Now on the circular arc (2.1) yields $f = (2/\pi)\log\cos(\frac{1}{2}\sigma)$, which is real, so $f = \phi$ there. Differentiating this relation gives $\partial\phi/\partial\sigma = -(1/\pi)\tan(\frac{1}{2}\sigma)$. We can use this expression to write $\partial y/\partial\sigma = (\partial y/\partial\phi)(\partial\phi/\partial\sigma) = -[v/(u^2+v^2)]\,(1/\pi)\tan(\frac{1}{2}\sigma)$. Upon inserting this result into the differentiated form of the constant pressure condition, we obtain

$$\frac{\partial|\zeta|^2}{\partial\sigma} - \left(\frac{2}{\pi}\tan\frac{\sigma}{2}\right)\frac{v}{u^2+v^2} = 0 \quad \text{on} \quad t = \mathrm{e}^{\mathrm{i}\sigma}, \quad 0 \geqslant \sigma \geqslant -\pi. \tag{2.3}$$

We now set $t = \mathrm{e}^{\mathrm{i}\sigma}$ in (2.2) to get $\zeta(\mathrm{e}^{\mathrm{i}\sigma})$, and we substitute that expression into (2.3). We will use the resulting equation to determine the unknown coefficients a_n that occur in (2.2).

To do so, we truncate the infinite series in (2.2) after $N-1$ terms. We find the $N-1$ coefficients a_n and the constant b by collocation. Thus we introduce $N-1$ mesh points,

$$\sigma_I = -\pi\frac{I-\frac{1}{2}}{N-1} \quad (I = 1, \ldots, N-1). \tag{2.4}$$

By using (2.2) we obtain ζ and $\partial|\zeta|^2/\partial\sigma$ at $\sigma = \sigma_I$ in terms of the coefficients a_n and b. Upon substituting these expressions into (2.3) we obtain $N-1$ nonlinear algebraic equations for the N unknowns $\{a_n\}_{n=0}^{N-2}$ and b. An Nth equation is obtained by fixing the position of the sink. This is achieved by specifying either a value for b or a value for h/W. To fix h/W we evaluate h and W by integrating the identity

$$\frac{\partial x}{\partial\phi} + \mathrm{i}\frac{\partial y}{\partial\phi} = \frac{1}{\zeta} \tag{2.5}$$

along the real diameter and along the circle in the t-plane.

We solved the resulting system of N equations by Newton's method. Once this system was solved we obtained the shape of the free surface by integrating (2.5) numerically. In addition, we calculated the Froude number F based on the total depth at infinity, which is defined by

$$F = Q[g(H+W)^3]^{-\frac{1}{2}}. \tag{2.6}$$

From our choice of dimensionless variables, it follows that F is given by

$$F = [\zeta(-1)]^{\frac{3}{2}}. \tag{2.7}$$

The coefficients a_n were found to decrease rapidly. For example, $a_1 \approx 0.3$, $a_{10} \approx -2\times 10^{-5}$ and $a_{20} \approx -2\times 10^{-8}$ for $h/W = 0.74$. Most of the calculations were done with $N = 30$.

3. Discussion of results

The present problem is qualitatively similar to the flow over a weir which we have already studied (Vanden-Broeck & Keller 1987). In particular, the flux Q and the ratio h/W cannot be specified independently when h/W is sufficiently small.

Following our analysis of weir flows we define a 'discharge coefficient' $C(h/W)$ by the relation

$$Q = C\left(\frac{h}{W}\right) g^{\frac{1}{2}} h^{\frac{3}{2}}. \tag{3.1}$$

We assume that for h/W small, $C(h/W)$ can be expanded in a finite Taylor series:

$$C\left(\frac{h}{W}\right) = C(0) + \frac{h}{W}C'(0) + O\left(\frac{h}{W}\right)^2. \tag{3.2}$$

Upon substituting (3.1) into (2.6) and expanding for h/W small, we obtain

$$F = \left(\frac{h}{W}\right)^{\frac{3}{2}}\left\{C(0) + \frac{h}{W}\left[C'(0) - \tfrac{3}{2}C(0)\right] + O\left[\left(\frac{h}{W}\right)^2\right]\right\}. \tag{3.3}$$

Relation (3.3) is similar to the relation (2.6) we derived for weir flows (Vanden-Broeck & Keller 1987).

Tuck & Vanden-Broeck (1984) solved the sink problem numerically for $h/W = 0$. They found a unique solution characterized by $(2Q)^2(gh^3)^{-1} = 12.622$. This result, together with (3.1), implies

$$C(0) = 1.776. \tag{3.4}$$

In order to find $C'(0)$ and $C(h/W)$ for $h/W \neq 0$ we plotted our numerical results for $F(h/W)^{-\frac{3}{2}}$ versus h/W. For h/W small, the graph is close to a straight line of slope -2.4 which intersects the $F(h/W)^{-\frac{3}{2}}$ axis at 1.78. Therefore (3.3) implies that

$$C(0) \sim 1.78, \tag{3.5}$$

$$C'(0) - \tfrac{3}{2}C(0) \sim -2.4. \tag{3.6}$$

Relations (3.5) and (3.6) yield

$$C'(0) = 0.3. \tag{3.7}$$

We note that (3.5) agrees with the value (3.4) previously obtained by Tuck & Vanden-Broeck (1984).

In figure 3 we have plotted the values of F versus h/W and versus b. For $h/W < 0.70$, which corresponds to $b > 0.37$, the flow is subcritical, i.e. $F < 1$. For $h/W > 0.70$, corresponding to $b < 0.37$, the flow is supercritical, i.e. $F > 1$. As $h/W \to 0.76$, $F \to \infty$. These solutions exist only for $h/W < 0.76$.

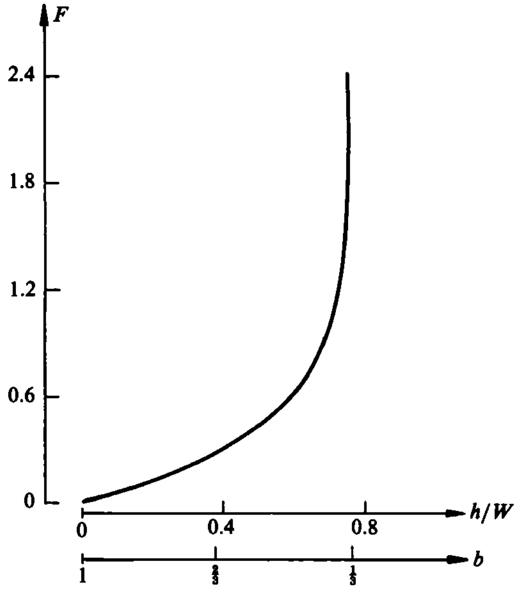

FIGURE 3. The Froude number F versus h/W and versus b.

As h/W increases from 0 to 0.76, the parameter b decreases monotonically from 1 to 0.33. We shall show later that this value is exactly $\frac{1}{3}$. A typical free-surface profile for $h/W = 0.5$ is shown in figure 1.

Our subcritical solutions for $h/W < 0.70$ are characterized by a uniform stream at infinity. This is exceptional since usually a subcritical flow has a train of waves at infinity. We believe that subcritical solutions with waves also exist for the present problem. Their wave amplitude a must be a function of F and h/W:

$$a = a(F, h/W). \tag{3.8}$$

The special expression (2.2) requires that there be no waves at infinity. Relation (3.8) indicates that this is possible only if some relation between F and h/W is satisfied, namely $a(F, h/W) = 0$. Such a relation is exactly what we discovered numerically (see figure 3).

The preceding considerations suggest that there are also additional supercritical solutions for $h/W > 0.70$. Supercritical flows are characterized by the presence of exponentially decreasing terms at infinity. Thus generally ζ has the form

$$\zeta \sim \zeta_\infty + A\,\mathrm{e}^{\pi\lambda f} \quad \text{as } \phi \to -\infty. \tag{3.9}$$

Here λ is the smallest positive root of

$$\pi\lambda - F^{-2}\tan\pi\lambda = 0. \tag{3.10}$$

Equation (3.10) has real solutions only for $F > 1$, i.e. only for supercritical flow. For $F < 1$, (3.10) has purely imaginary roots and then (3.9) corresponds to a train of small amplitude waves at infinity. Therefore the exponentially small term which occurs in the expression for supercritical flows is the 'analytic continuation' of the small-amplitude wave term of subcritical flows. As in the subcritical case, the factor A in (3.9) must depend upon F and h/W:

$$A = A(F, h/W). \tag{3.11}$$

In terms of the transformation (2.1) from f to t, we can rewrite (3.9) as

$$\zeta \sim \zeta_\infty + A(t+1)^{2\lambda} \quad \text{as } t \to -1. \tag{3.12}$$

The special expression (2.2) does not allow a singularity like that in (3.12) as $t \to -1$. Therefore, the supercritical solutions we have computed correspond to $A(F, h/W) = 0$. This agrees with our numerical results which show that solutions of the form (2.2) can be obtained only if some relation between F and h/W is satisfied, as figure 3 shows.

4. Additional supercritical flows

We shall now construct solutions for which $A(F, h/W) \neq 0$. For $F = \infty$, the velocity is constant on the free surface and the problem has an exact solution, namely

$$\zeta(t) = -\mathrm{i}\frac{(t+b)^{\frac{1}{2}}}{t(1+tb)^{\frac{1}{2}}}\zeta(-1). \tag{4.1}$$

Solution (4.1) is defined for all values of b. By using (2.1) and (4.1) we find that

$$\zeta \sim \zeta(-1) + \mathrm{e}^{\frac{1}{2}\pi\phi/2}\frac{1-3b}{2-2b}\zeta(-1) \quad \text{as } \phi \to -\infty. \tag{4.2}$$

Relation (3.10) shows that $\lambda \to \frac{1}{2}$ as $F \to \infty$. Therefore matching (3.9) and (4.2), and using (2.7), yields

$$A \sim F^{\frac{2}{3}}\frac{1-3b}{2-2b} \quad \text{as } F \to \infty. \tag{4.3}$$

The asymptotic expression (4.3) vanishes for $b = \frac{1}{3}$. The corresponding value of h/W is 0.76. This is the solution for $F = \infty$ obtained by using (2.2). (See figure 3). However solutions for $F = \infty$ exist for $b \neq \frac{1}{3}$. They are given by (4.1).

Solutions for F large can be obtained by using perturbation theory: (4.1) can be considered as the first term of an expansion in powers of F^{-1}. Instead of doing that, we shall solve the problem numerically for arbitrary values of F.

We generalize (2.2) to allow the singularity (3.12) at $t = -1$. Therefore we write

$$\zeta = \mathrm{i}(t+b)^{\frac{1}{2}}t^{-1}\exp\left[A(1+t)^{2\lambda} + \sum_{n=0}^{\infty} a_n t^n\right]. \tag{4.4}$$

Here λ is the smallest positive root of (3.10). The coefficients a_n and the constant A are to be found. We truncate the infinite series after $N-1$ terms and, as before, satisfy (2.3) at the collocation points (2.4). This leads to $N-1$ algebraic equations for the $N+1$ unknowns $\{a_n\}_{n=0}^{N-2}$, b and A. Two more equations are obtained by specifying F and the position of the sink, i.e. h/W or b. The resulting system of $N+1$ equations with $N+1$ unknowns is solved by Newton's method.

We note that we specify two parameters F and h/W or b, in order to obtain a unique solution. We checked numerically that specifying more or less parameters did not yield convergence for N large.

In figure 4 we present numerical values of the parameter $A/F^{\frac{2}{3}}$ appearing in (4.4) versus F for $b = 0.3, 0.35$ and 0.4. As $F \to \infty$, the curves approach asymptotically the values (4.3). The points at which the curves intersect the F-axis correspond to supercritical solutions for which $A = 0$ in (4.4). These are the solutions described by the portion $\frac{1}{3} < b < 0.37$ of the curve in figure 3.

For $b < \frac{1}{3}$ or $b > 0.37$ the curves in figure 4 do not intersect the F-axis. For $b > 0.37$ we found that as the Froude number is decreased from infinity each curve in figure 4 reaches a limit point and then turns back. Therefore for some values of the Froude number two different solutions are possible. A typical profile for $b = 0.4$, $F = 1.3$ and

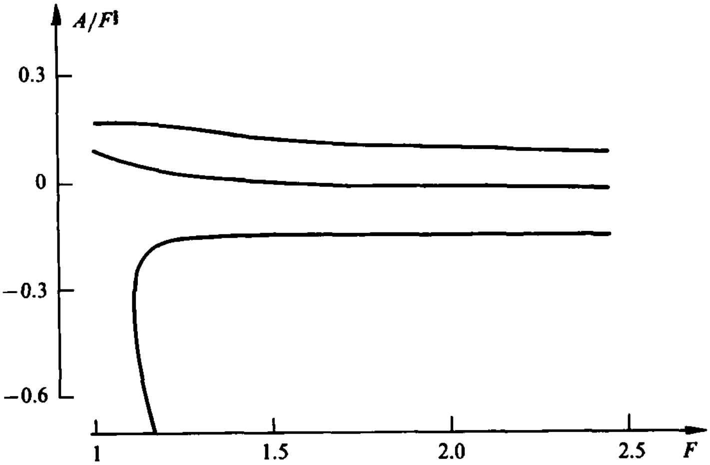

FIGURE 4. Computed values of $AF^{-\frac{1}{2}}$ versus F for additional supercritical flows with $b = 0.3$ (top curve), $b = 0.35$ (middle) and $b = 0.4$ (bottom).

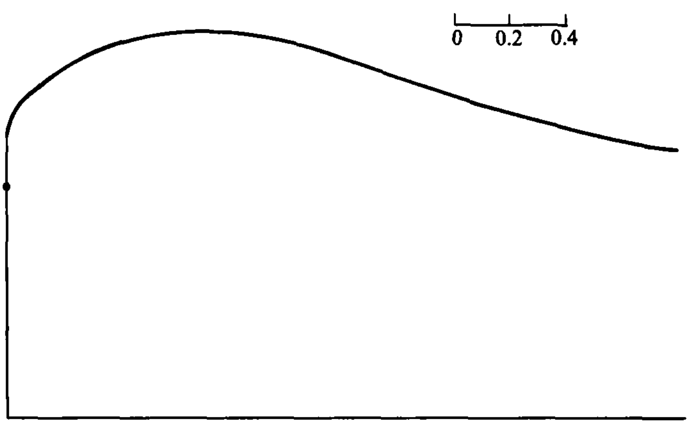

FIGURE 5. The free surface profile of one of the family of additional supercritical flows with $b = 0.4$, $F = 1.3$ and $AF^{-\frac{1}{2}} = -1.15$. The dot indicates the position of the source.

$AF^{-\frac{1}{2}} = -1.15$ is shown in figure 5. We expect that each solution branch for $b \leqslant 0.37$ will end when a stagnation point appears on the free surface with a 120° angle at it.

For $b < 0.37$ the curves of figure 4 extend from $F = \infty$ to $F = 1_+$. We did not continue these branches into the range $F < 1$. Since $A \neq 0$ as $F \rightarrow 1_+$, we expect (by analytic continuation) the solutions for $F < 1$ to have waves at infinity, i.e. to have $a \neq 0$ in (3.8).

The results above show that there are two different classes of supercritical flows. The first class is characterized by the fact that subcritical flows are ultimately reached when the Froude number is decreased from infinity (the two top curves in figure 4). In the second class a limit point is reached before the subcritical regime is reached (the bottom curve in figure 4). Another example of supercritical flow of the second class has been found by Vanden-Broeck (1986) in his analysis of flows under a gate.

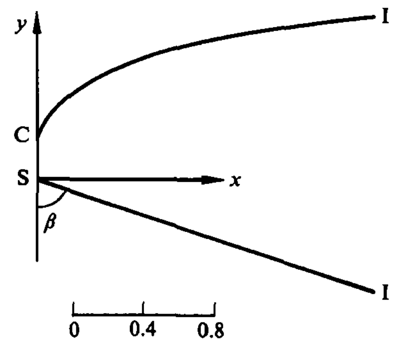

FIGURE 6. Vertical section of a reservoir with a sloping bottom and a vertical wall. The sink S is at the corner. The free surface is CI, and β is the angle between the bottom and the vertical. The figure is an actual profile for $\beta/\pi = 0.4$. The vertical scale is the same as the horizontal scale.

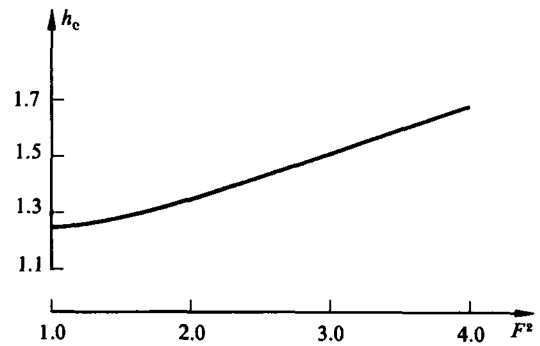

FIGURE 7. Cusp location h_c versus F^2 for $\beta = \frac{1}{2}\pi$. The particular solution obtained in the limit as $\beta \to \frac{1}{2}\pi$ corresponds to $F = 1$.

5. Sink on a sloping bottom

We shall now extend the procedure of §2 to obtain the flow in the region of the z-plane shown in figure 6. The rigid bottom slopes at an angle β from the vertical. At the origin there is a sink of strength $2\pi Q/(\pi-\beta)$. We choose $\psi = 0$ on the vertical wall and on the free surface. It follows from the value of the sink strength that $\psi = Q$ on the sloping bottom. As before we choose the unit of length and the unit of velocity so that $Q = 1$ and $g = 1$. By using the transformation (2.1) we map the flow domain into the interior of the unit-circle in the t-plane (see figure 2).

Since there is a sink of strength $2\pi Q/(\pi-\beta)$ at $t = 0$ and a source at $t = -1$, ζ must have singularities at these two points and be regular elsewhere. The appropriate singularities are proportional to $t^{\gamma-1}$ at $t = 0$ and to $(1+t)^{1-2\gamma}$ at $t = -1$, where $\gamma = \beta/\pi$. Therefore we write

$$\zeta = \mathrm{i}(1+t)^{1-2\gamma}\, t^{\gamma-1} \exp\left[\sum_{n=0}^{\infty} a_n t^n\right]. \tag{5.1}$$

The coefficients a_n are found by following the numerical procedure of §2.

We shall consider first the special case $\beta = \frac{1}{3}\pi$ corresponding to $\gamma = \frac{1}{3}$. Then by choosing $a_0 = \frac{1}{3}\log(3/\pi)$ and $a_n = 0$ for $n \geqslant 1$ we obtain an exact solution of (2.3). This is the solution found by Craya (1949). For $\beta = 0$ (i.e. $\gamma = 0$) the problem reduces to the flow calculated by Tuck & Vanden-Broeck (1984). Hocking (1985) obtained

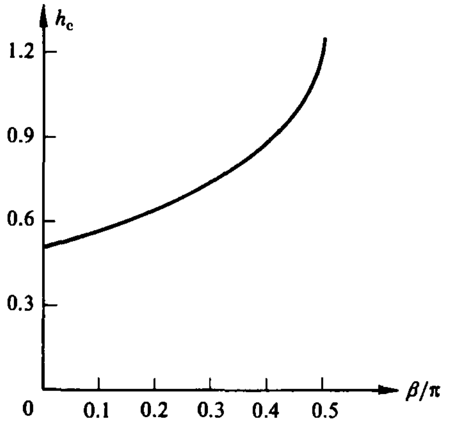

FIGURE 8. Cusp depth h_c versus β/π for the solutions found by Hocking. Craya's solution is at $\beta = \frac{1}{3}\pi$ and that of Tuck & Vanden-Broeck is at $\beta = 0$.

numerical solutions for a sequence of angles ranging from 0 to $\frac{1}{2}\pi$. Our numerical results were found to be in good agreement with theirs. In particular we confirmed that a unique solution exists for each value of $0 \leqslant \beta < \frac{1}{2}\pi$. All these solutions are characterized by a stagnation point at infinity. For $\beta = \frac{1}{2}\pi$ there is a uniform stream at infinity. Our numerical results show that a unique solution characterized by $F = 1$ is obtained as $\beta \rightarrow \frac{1}{2}\pi$. However there is a family of solutions for $\beta = \frac{1}{2}\pi$. This family can be computed by choosing $b = 0$ in (4.4) and using the numerical procedure of §2. As $F \rightarrow 1$, $\lambda \rightarrow 0$. Therefore the limit as $F \rightarrow 1$ of (4.4) with $b = 0$, agrees with the limit of (5.1) as $\gamma \rightarrow \frac{1}{2}$. The coefficients a_n, $n > 1$ in both expressions become identical and the difference between the coefficients a_0 is equal to A. These results are illustrated in figures 7 and 8. In figure 7 we present numerical values of the depth h_c of the cusp under the level of the free surface at infinity (see figure 6) versus F^2 for $\beta = \frac{1}{2}\pi$. In figure 8 we present numerical values of h_c versus β for $0 \leqslant \beta < \frac{1}{2}\pi$.

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REFERENCES

COLLINGS, I. L. 1986 *J. Austral. Math. Soc.* B **28**, 260.

CRAYA, A. 1949 *Houille Blanche* **4**, 44.

HOCKING, G. C. 1985 *J. Austral. Math. Soc.* B:**26**, 470.

TUCK, E. O. & VANDEN-BROECK, J.-M. 1984 *J. Austral. Math. Soc.* B:**25**, 443.

VANDEN-BROECK, J.-M. 1986 *Phys. Fluids* **29**, 3148.

VANDEN-BROECK, J.-M. & KELLER, J. B. 1987 *J. Fluid Mech.* **176**, 283.

YIH, C. S. 1965 *Dynamics of Nonhomogeneous Fluids*. Macmillan.